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RIEMANNIAN ALMOST PRODUCT MANIFOLDS GENERATED BY A CIRCULANT STRUCTURE

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Abstract: *A 4-dimensional Riemannian manifold equipped with a circulant structure, which is an isometry with respect to the metric and its fourth power is the identity, is considered. The almost product manifold associated with the considered manifold is studied. The relation between the covariant derivatives of the almost product structure and the circulant structure is obtained. The conditions for the covariant derivative of the circulant structure, which imply that an almost product manifold belongs to each of the basic classes of the Staikova-Gribachev classification, are given.*

Key words: *Riemannian metric, circulant matrix, almost product structure.*

Mathematics Subject Classification (2010). *Primary 53B20, 53C15, Secondary 15B05.*

1. Introduction

The circulant matrices and the circulant structures have application to Vibration analysis, Graph theory, Linear codes, Geometry (for example [1], [2] and [3]). Riemannian manifolds equipped with a circulant structure, whose fourth power is the identity were considered in [4] and [5]. In particular case, such manifolds could be Riemannian almost product manifolds. The systematic development of the theory of Riemannian manifolds M with a metric g and an almost product structure P was started by K. Yano in [6]. In [7] A. M. Naveira classified the almost product manifolds (M,P,g) with respect to the covariant derivative of P. The Riemannian almost product manifolds (M,P,g) with zero trace of the structure P were classified with respect to the covariant derivative of P by M. Staikova and K. Gribachev in [8]. The basic classes in this classification are W_1, W_2 and W_3. The class $W_0 = W_1 \cap W_2 \cap W_3$ was called the class of Riemannian P-manifolds. Our purpose is to obtain characteristic conditions for each of these classes according to the circulant structure. In the present paper we consider a 4-dimensional differentiable manifold M with a Riemannian metric g and a circulant structure Q, whose fourth power is the identity and Q acts as an isometry on g. This manifold we will denote by (M,Q,g). We study the Riemannian almost product manifold (M,P,g) where $P = Q^2$. The paper is organized as follows. In Sect. 1, some necessary facts about considered manifolds (M,Q,g) and (M,P,g) are recalled. In Sect. 2, the relation between the covariant derivative of P and the covariant derivative of Q is obtained. In Sect. 3, the conditions for the covariant derivative of Q, which imply that (M,P,g) belongs to each of the basic classes of the Staikova-Gribachev classification, are given.

2. Preliminaries

Let M be a 4-dimensional Riemannian manifold equipped with a metric g and an endomorphism Q in the tangent space T_pM at an arbitrary point p on M. Let the coordinates of Q with respect to some basis $\{e_i\}$ of T_pM form the circulant matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{1.1}$$

Then Q satisfies the equalities

$$Q^4 = id, \ Q^2 \neq \pm id.$$

Let the structure Q be compatible with the metric g, i.e.

$$g(Qx, Qy) = g(x, y). \tag{1.2}$$

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Here and anywhere in this work x, y, z, u will stand for arbitrary elements of the algebra of the smooth vector fields on M or vectors in T_pM. The Einstein summation convention is used, the range of the summation indices being always $\{1, 2, 3, 4\}$.

Further, we consider a manifold (M, Q, g) equipped with a metric g and a structure Q, which satisfy (1.1) and (1.2). This manifold is studied in [4] and [5].

We denote $P = Q^2$. In [4] it is noted that the manifold (M, P, g) is a Riemannian manifold with an almost product structure P, because $P^2 = id$, $P \neq \pm id$ and $g(Px, Py) = g(x, y)$. Moreover $trP = 0$. For such manifolds is valid the Staikova-Gribachev classication given in [8]. This classification was made with respect to the tensor F of type $(0,3)$ and the Lee form α, which are defined by

$$F(x, y, z) = g((\nabla_x P)y, z),$$
$$\alpha(x) = g^{ij} F(e_i, e_j, x). \tag{1.3}$$

Here ∇ is the Levi-Civita connection of g, and g^{ij} are the components of the inverse matrix of g with respect to $\{e_i\}$.

The basic classes of the Staikova-Gribachev classification are W_1, W_2 and W_3. Their intersection is the class of Riemannian P-manifolds W_0. A manifold (M, P, g) belongs to each of these classes if it satisfies the following conditions:

$$W_0: \quad F(x, y, z) = 0, \tag{1.4}$$

$$W_1: \quad F(x, y, z) = \frac{1}{4}((g(x, y)\alpha(z) + g(x, z)\alpha(y) - g(x, Py)\alpha(Pz) - g(x, Pz)\alpha(Py)), \tag{1.5}$$

$$W_2: \quad F(x,y,Pz) + F(y,z,Px) + F(z,x,Py) = 0, \quad \alpha(z) = 0, \tag{1.6}$$

$$W_3: \quad F(x, y, z) + F(y, z, x) + F(z, x, y) = 0. \tag{1.7}$$

It is well known that ∇ satisfies the equalities:

$$(\nabla_x Q)y = \nabla_x Qy - Q\nabla_x y \tag{1.8}$$

$$(\nabla_x P)y = \nabla_x Py - P\nabla_x y \tag{1.9}$$

Let the structure Q of a manifold (M, Q, g) be the covariant constant, i.e. $(\nabla_x Q)y = 0$. Then, from (1.8) we obtain successively $\nabla_x Qy = Q\nabla_x y$, $\nabla_x Q^2 y = Q\nabla_x Qy = Q^2 \nabla_x y$, thus we get $\nabla_x Py = P\nabla_x y$.
Therefore, from (1.9) it follows $(\nabla_x P)y = 0$.
By using the latter equality and (1.3) we find (1.4). Hence the next theorem is valid.

Theorem 1.1. If the structure Q of the manifold (M, Q, g) satisfies $\nabla Q = 0$, then (M, P, g) belongs to the class W_0.

As it is known the curvature tensor R of ∇ is determined by

$$R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z.$$

The corresponding tensor of type $(0,4)$ is defined as follows $R(x, y, z, u) = g(R(x, y)z, u)$.

Proposition 1.2. [5] If the structure Q of the manifold (M, Q, g) satisfies $\nabla Q = 0$, then for the curvature tensor R it is valid

$$R(x, y, Qz, Qu) = R(x, y, z, u).$$

We substitute Qz for z and Qu for u in the latter equality, and using Theorem 1.1, we obtain

Corollary 1.3. If the manifold (M, P, g) belongs to W_0, then the curvature tensor R satisfies,

$$R(x, y, Pz, Pu) = R(x, y, z, u).$$

i.e. R is a Riemannian P-tensor.

3. Relation between F and $\overline{F}$

We consider manifolds (M, Q, g) and (M, P, g), where $P = Q^2$. We define a tensor $\overline{F}$ of type $(0,3)$, as follows

$$\overline{F}(x, y, z) = g((\nabla_x Q)y, z),$$
$$\overline{\alpha}(x) = g^{ij} \overline{F}(e_i, e_j, x), \tag{2.1}$$

where $\overline{\alpha}$ is the Lee form associated to $\overline{F}$.

Theorem 2.1. For the tensors F on the manifold (M,P,g) and $\overline{F}$ on the manifold (M,Q,g) the following equalities are valid:

$$\overline{F}(x,y,z)+\overline{F}(x,Qy,Qz)=F(x,y,Qz), \quad (2.2)$$

$$\overline{F}(x,y,Q^3z)+\overline{F}(x,Qy,z)=F(x,y,z). \quad (2.3)$$

Proof. From (1.3) and (1.9), due to $P=Q^2$, we get

$$F(x,y,z)=g(\nabla_x Py-P\nabla_x y,z) = g(\nabla_x Q^2 y-Q^2\nabla_x y,z),$$

i.e. $F(x,y,z)=g(\nabla_x Q^2 y-Q^2\nabla_x y,z)$.
Then

$$F(x,y,Qz)=g(\nabla_x Q^2 y,Qz)-g(Q^2\nabla_x y,Qz),$$

from which, because of (1.2) we have

$$F(x,y,Qz)=g(\nabla_x Q^2 y,Qz) - g(Q\nabla_x y,z), \quad (2.4)$$

From (1.8) and (2.1) we obtain

$$\overline{F}(x,y,z)=g(\nabla_x Qy,z) - g(Q\nabla_x y,z), \quad (2.5)$$

and consequently

$$\overline{F}(x,Qy,Qz)=g(\nabla_x Q^2 y,Qz) - g(\nabla_x Qy,z). \quad (2.6)$$

Taking the sum of (2.5) and (2.6) we get

$$\overline{F}(x,y,z)+\overline{F}(x,Qy,Qz) = g(\nabla_x Q^2 y,Qz)-g(Q\nabla_x y,z).$$

Then, having in mind (2.4), we find (2.2). Now we substitute Q^3z for z into (2.2) and using (1.2), we find (2.3). Hence the next theorem is valid.

Theorem 2.2. The manifold (M,P,g) belongs to W_0 if and only if Q satisfies

$$(\nabla_x Q)Qy=-Q(\nabla_x Q)y. \quad (2.7)$$

Proof. Let $(M,P,g)\in W_0$, i.e. $F=0$. Then, due to (2.2), it follows

$$\overline{F}(x,Qy,Qz)=-\overline{F}(x,y,z).$$

The latter equality and (2.1) imply (2.7).

Vice versa. According to (2.1) and (2.7) we find $\overline{F}(x,y,z)+\overline{F}(x,Qy,Qz)=0$.
Then, due to (2.2), it follows $F=0$, i.e. the manifold $(M,P,g)\in W_0$.

4. Properties of $\overline{F}$

Theorem 3.1. For the tensor $\overline{F}$ on (M,Q,g) the following equalities are valid:

$$\overline{F}(x,y,Q^3z)+\overline{F}(x,Qy,z) = \overline{F}(x,z,Q^3y)+\overline{F}(x,Qz,y), \quad (3.1)$$

$$\overline{F}(x,y,z)+\overline{F}(x,Qy,Qz) + \overline{F}(x,Q^2y,Q^2z)+\overline{F}(x,Q^3y,Q^3z)=0, \quad (3.2)$$

$$\overline{F}(x,y,Qz)=-\overline{F}(x,z,Qy), \quad (3.3)$$

$$\overline{F}(x,y,Q^3z)=-\overline{F}(x,Q^2z,Qy). \quad (3.4)$$

Proof. It is known that the tensor F determined by (1.3) has the properties:

$$F(x,y,z)=F(x,z,y), \quad (3.5)$$

$$F(x,Py,Pz)=-F(x,y,z). \quad (3.6)$$

Equalities (2.3) and (3.5) imply (3.1).
From (2.3) and (3.6) we get

$$\overline{F}(x,Qy,z)+\overline{F}(x,y,Q^3z) + \overline{F}(x,Q^3y,Q^2z)+\overline{F}(x,Q^2y,Qz)=0.$$

In the latter equality we substitute Qz for z, and we obtain (3.2). Further, we substitute Qz for z into (2.5) and we have

$$\begin{aligned}\overline{F}(x,y,Qz) &= g(\nabla_x Qy, Qz) - g(\nabla_x y, z)\\ &= xg(Qy,Qz) - g(Qy, \nabla_x Qz)\\ &\quad - xg(y,z) + g(y, \nabla_x z)\\ &= -g(\nabla_x Qz, Qy) + g(\nabla_x z, y)\\ &= -g(\nabla_x Qz, Qy) + g(Q\nabla_x z, Qy)\\ &= -g(\nabla_x Qz - Q\nabla_x z, Qy) = -\overline{F}(x,z,Qy).\end{aligned}$$

Therefore we get (3.3). From (3.3) directly follows (3.4). Using (1.5), (2.1) and (2.2) we obtain the following

Theorem 3.2. The manifold (M,P,g) belongs to W_1 if and only if the tensor $\overline{F}$ on (M,Q,g) satisfies the following conditions:

$$\begin{aligned}&\overline{F}(x,y,Q^3 z) + \overline{F}(x,Qy,z)\\ &= \frac{1}{4}(g(x,y)\alpha(z) + g(x,z)\alpha(y)\\ &+ g(x,Q^2 y)\alpha(Q^2 z) + g(x,Q^2 z)\alpha(Q^2 y),\end{aligned}$$

$$\overline{\alpha}(Q^3 z) + g^{ij}\overline{F}(e_i, Qe_j, x) = \alpha(z).$$

We apply (2.2) and (2.3) into (1.6) and we find

$$F(x,y,Pz) = \overline{F}(x,y,Qz) + \overline{F}(x,Qy,Q^2 z).$$

Therefore we arrive at the following

Theorem 3.3. The manifold (M,P,g) belongs to W_2 if and only if the tensor $\overline{F}$ on (M,Q,g) satisfies the following condition

$$\begin{aligned}&\overline{F}(x,y,Qz) + \overline{F}(x,Qy,Q^2 z)\\ &+ \overline{F}(y,z,Qx) + \overline{F}(y,Qz,Q^2 x)\\ &+ \overline{F}(z,x,Qy) + \overline{F}(z,Qx,Q^2 y) = 0.\end{aligned}$$

We apply (2.3) into (1.7) and we have

Theorem 3.4. The manifold (M,P,g) belongs to W_3 if and only if the tensor $\overline{F}$ on (M,Q,g) satisfies the following condition

$$\begin{aligned}&\overline{F}(x,y,Q^3 z) + \overline{F}(x,Qy,z)\\ &+ \overline{F}(y,z,Q^3 x) + \overline{F}(y,Qz,x)\\ &+ \overline{F}(z,x,Q^3 y) + \overline{F}(z,Qx,y) = 0.\end{aligned}$$

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